Kinematics of a Rigid Body

Definition of Rigid Body: A system of particles for which the distances between particles remain unchanged.

Kinematics: Study of motion without considering force.

What is the difference between particle motion and rigid body motion?
- Particle: only translational motion
- Rigid body: translational and rotational motion

How to describe rigid body motion?
- Need body fixed reference frame to describe translational and rotational motion.
Rigid Body Motion

Translational motion

Path of rectilinear translation

Path of curvilinear translation

Figure 16-01a

Figure 16-01b
Rigid Body Motion

Rotational motion

General motion = translational + rotational motion

Rotation about a fixed axis

General plane motion

Figure 16-01c

Figure 16-01d
Rigid Body Motion

General plane motion

Curvilinear translation

Rectilinear translation

Rotation about a fixed axis

Figure 16-02
Translation

position \( \mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \)

velocity \( \mathbf{v}_B = \mathbf{v}_A \)

acceleration \( \mathbf{a}_B = \mathbf{a}_A \)
Rotation About a Fixed Axis

Angular motion

Angular position: at the instant shown, the angular position of \( r \) is defined by the angle \( \theta \) measured between a fixed reference line and \( r \).

Angular displacement: the change in the angular position, which can be measured as a differential \( d\theta \)
Rotation About a Fixed Axis

Angular velocity
\[ \omega = \frac{d \theta}{dt} \quad \text{Unit: rad/sec} \]

Angular Acceleration
\[ \alpha = \frac{d \omega}{dt} \quad \alpha = \frac{d^2 \theta}{dt^2} \quad \alpha d \theta = \omega d \omega \]

Constant Angular Acceleration
\[ \omega = \omega_0 + \alpha_c t \]
\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha_c t^2 \]
\[ \omega^2 = \omega_0^2 + 2 \alpha_c (\theta - \theta_0) \]
Rotation About a Fixed Axis

Motion of point P

Position vector $\mathbf{r}$

velocity

$v = \omega \times \mathbf{r}$

$v = \omega r$

$v = \frac{d\mathbf{r}}{dt} = \omega \times \mathbf{r}$
Rotation About a Fixed Axis

Acceleration

\[ a_t = \alpha r \]
\[ a_n = \omega^2 r \]
\[ a = \alpha \times r + \omega \times (\omega \times r'_p) \]
\[ a = a_t + a_n \]
\[ = \alpha \times r - \omega^2 r \]

\[ a = \frac{dv}{dt} = \frac{d}{dt} (\omega \times r) = \frac{d\omega}{dt} \times r + \omega \times \frac{dr}{dt} \]
\[ = \alpha \times r + \omega \times (\omega \times r) \]
The two pulleys are operated by the tape which passes over them without slipping. During start-up the speed \( v \) of the tape increases from 1.2 m/s to 2.4 m/s with constant acceleration while 600 mm of tape have passed over the pulleys. When the speed approaches 2.4 m/s, determine the angular acceleration \( \alpha_A \) of pulley A and the magnitude \( a_p \) of the total acceleration of point P fixed to pulley B.
The horizontal control rod has a constant velocity $v = 0.2 \text{ m/s}$. Calculate the angular velocity $\omega$ of rod AB when $x = 300 \text{ mm}$. 
Relative Motion Analysis: Velocity

\[ \mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \]

Figure 16-10a
Relative Motion Analysis: Velocity

Displacement

\[ \mathbf{dr}_B = \mathbf{dr}_A + \mathbf{dr}_{B/A} \]

Figure 16-10b

Figure 16-10c

Translation
Rotation
Relative Motion Analysis: Velocity

Velocity

\[
\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt}
\]

\[
\frac{d\mathbf{r}_{B/A}}{dt} = \frac{r_{B/A}}{dt} \frac{d\theta}{dt} = r_{B/A} \omega
\]

\[
\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}
\]

\(\mathbf{v}_B\) = velocity of Point B

\(\mathbf{v}_A\) = velocity of the base Point A

\(\mathbf{v}_{B/A}\) = relative velocity of "B with respect to A"

This relative motion is circular the magnitude is 

\[
\mathbf{v}_{B/A} = \omega r_{B/A} \text{ and the direction is perpendicular to } \mathbf{r}_{B/A}
\]
Relative Motion Analysis: Velocity

\[ \mathbf{v}_B = \mathbf{v}_A + \omega \times \mathbf{r}_{B/A} \]

- \( \mathbf{v}_B \): velocity of \( B \)
- \( \mathbf{v}_A \): velocity of the base Point \( A \)
- \( \omega \): angular velocity of the body
- \( \mathbf{r}_{B/A} \): relative position vector drawn from \( A \) to \( B \)
Relative Motion Analysis: Velocity

Figure 16-11b
Relative Velocity Example

At the instant shown for the 4-bar linkage, $\theta = \tan \frac{3}{4}$ and $\dot{\theta} = \frac{4}{4}$ rad/s. Determine the corresponding angular velocities of $AB$ and $CB$. Dimensions in millimeters.
From geometry

\[ \frac{V_B}{V_A} = \frac{V_B}{V_A} \]
Relative Velocity Example

Crank OA oscillates about the $\theta = 0$ position causing CB to oscillate in turn. If OA has a counterclockwise angular velocity of 6 rad/s when $\theta = 30^\circ$, determine the corresponding angular velocity of CB.